HW 03: Random Variables

> ❗ Due date
>
> This assignment is due on **Wednesday, October 15 before class (hard deadline so I will discuss the solution in class for review**. To be considered on time, the following must be done by the due date:
>
> - Final `.pdf` file submitted on Gradescope

Learning goals

In this assignment, you will develop the ability to:

- **Master Discrete & Continuous Distributions:** Calculate **probabilities, means, and variances** for both discrete (Binomial, Poisson, Joint PMFs) and continuous (Uniform, Exponential, General PDFs) random variables.
- **Analyze Dependence and Transformations:** Compute **marginal and conditional distributions** from joint PMFs, and find the **expected values** of transformed variables (e.g., Lognormal and Chi-Square).
- **Apply Distributional Properties:** Use key properties like the **memoryless property** (Exponential) and the **Central Limit Theorem** connections to solve practical problems and approximations.

Conceptual exercises

Instructions

The conceptual exercises focus on explaining concepts and showing results mathematically. Show your work for each question. I expect you not to skip any steps and carefully understand the thinking process when working on these exercises.

You may write the answers and associated work for conceptual exercises by hand or type them in your Quarto document.

Problem 1 (Discrete pmf and expectation)

A survey records the number of streaming subscriptions in 3 independent households. Let S_i be the subscriptions in household i, where $i \in \{1, 2, 3\}$. Each household has the following probability mass function (pmf):

$$P(S_i = 0) = 0.5, \quad P(S_i = 1) = 0.3, \quad P(S_i = 2) = 0.2.$$

Let $X = S_1 + S_2 + S_3$ be the total subscriptions across 3 independent households.

1. Write $p_X(x)$ in terms of p_0, p_1, p_2 where $p_j = P(S_i = j)$.
2. Compute $P(X \leq 2)$.
3. Compute $E[X]$ and $\text{Var}(X)$.

Problem 2 (Binomial and Poisson)

A retailer has $n = 400$ visitors per day. Each makes a purchase with probability $p = 0.01$, independently of others.

1. Identify the distribution of the number of purchase per day Y, write its pmf, and find $E[Y]$ and $\text{Var}(Y)$.
2. Write an expression for $P(Y \geq 8)$.
3. Use a Poisson approximation to compute $P(Y \geq 8)$.

Problem 3 (Joint pmf and conditional)

The joint pmf of cars X and bicycles Y owned by a household is given by the table below:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.08	0.02
$X = 1$	0.20	0.35	0.05
$X = 2$	0.01	0.07	0.12

Note: The marginal totals have been added for convenience, but the original problem did not include them.

1. Find $p_X(x)$ and $p_Y(y)$.
2. Compute $P(X \geq 1, Y \leq 1)$.
3. Compute the conditional pmf $p_{X|Y}(x|y = 1)$.
4. Compute $E[X|Y = 1]$ and $\text{Var}(X|Y = 1)$.

Problem 4 (Continuous pdf)

The waiting time (months) for microloan approval has the probability density function (pdf):

$$f_T(t) = \begin{cases} \alpha t, & 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the constant α.
2. Find the Cumulative Distribution Function (CDF) $F_T(t)$.
3. Compute the probability $P(0.5 \le T \le 1.5)$.
4. Find the expected value $E[T]$ and the variance $\mathrm{Var}(T)$.

Problem 5 (Normal, Lognormal, Chi-square)

1. If $Z \sim N(0, 1)$, what distribution does $W = Z^2$ follow? Give its mean and variance.
2. If $Y \sim N(3, 0.25)$, let $X = e^Y$. Find the expected value $E[X]$ and interpret the result.
3. If $Z_1, Z_2, Z_3 \sim$ i.i.d. $N(0, 1)$, find the distribution of $S = \sum Z_i^2$ and compute $P(S > 7)$.

Problem 6 (Exponential)

Let $X \sim \text{Exponential}(\theta = 0.2)$, where X is time between arrivals (days).

1. Find $E[X]$ and $\mathrm{Var}(X)$.
2. Compute the conditional probability $P(X > 10 \mid X > 5)$.

Grading (30 points)

Choose 3 problem out of 6

Component	Points
Option 1	10
Option 2	10
Option 3	10

"Formatting" grade is to assess whether or not you mark specific page of the pdf to a specific question to help the TA when grading.