AE Lecture 9 Solution

Continuous Random Variables

Applied Exercise 1

Problem: A random variable $X \sim \text{Uniform}(0, 2)$. Compute the population mean, standard deviation, the quantile function, and the median (quantiles at $q = 0.5$).

Solution

Finding the constant c for the PDF:

For a uniform distribution on $[0, 2]$, the pdf has the form:

$$f_X(x) = \begin{cases} c, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

For a valid pdf, we need $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$:

$$\begin{aligned} \int_0^2 c\, dx &= 1 \\ c[x]_0^2 &= 1 \\ c(2-0) &= 1 \\ 2c &= 1 \\ c &= \frac{1}{2} \end{aligned}$$

Therefore:

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Population Mean (using integral):

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 x \cdot \frac{1}{2}\,dx \\
&= \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 \\
&= \frac{1}{2}\cdot\frac{4}{2} = \frac{1}{2}\cdot 2 = 1
\end{aligned}
$$

Variance (Method 1 - Direct Integration):

$$
\begin{aligned}
\mathrm{Var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx = \int_0^2 (x-1)^2 \cdot \frac{1}{2}\,dx \\
&= \frac{1}{2}\int_0^2 (x^2 - 2x + 1)\,dx \\
&= \frac{1}{2}\left[\frac{x^3}{3} - x^2 + x\right]_0^2 \\
&= \frac{1}{2}\left(\frac{8}{3} - 4 + 2\right) = \frac{1}{2}\left(\frac{8}{3} - 2\right) \\
&= \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}
\end{aligned}
$$

Variance (Method 2 - Using $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$**):**

First, compute $E(X^2)$:

$$
\begin{aligned}
E(X^2) &= \int_0^2 x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 \\
&= \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 = \frac{4}{3} - 1^2 \\
&= \frac{4}{3} - 1 = \frac{1}{3}
\end{aligned}
$$

Standard Deviation:

$$
\sigma_X = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577
$$

CDF:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x-0}{2-0} = \frac{x}{2}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

Quantile Function: For $0 < q < 1$, solve $F_X(\tau_{X,q}) = q$:

$$\frac{\tau_{X,q}}{2} = q \implies \tau_{X,q} = 2q$$

Median:

$$\text{Median} = \tau_{X,0.5} = 2(0.5) = 1$$

PDF and CDF for Exercise 1

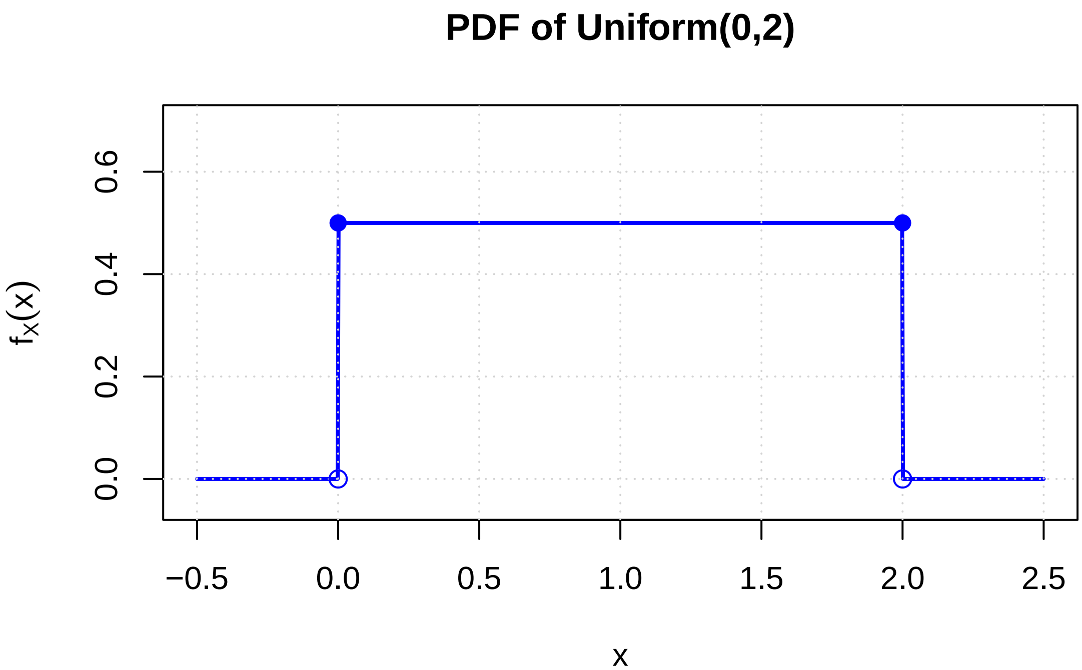

Figure 1: PDF of Uniform(0,2) Distribution

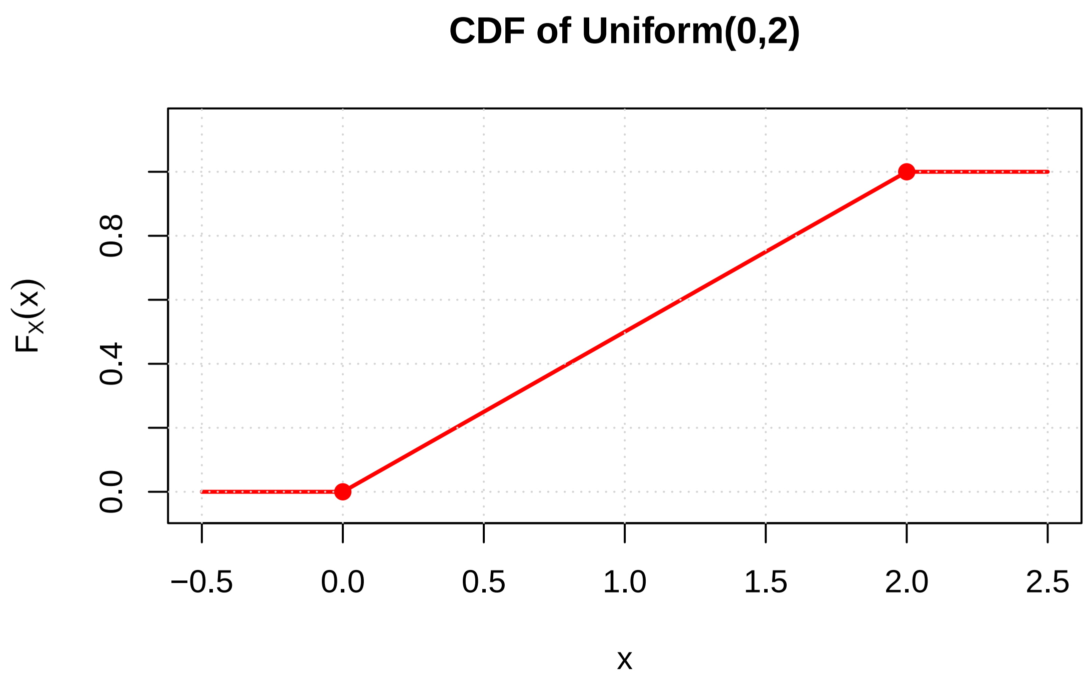

Figure 2: CDF of Uniform(0,2) Distribution

Applied Exercise 2

Problem: X is a continuous random variable with the following pdf:

$$f_X(x) = \begin{cases} ax, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of a, the cdf $F_X(x)$, the quantile function $\tau_{X,q}$, and the population mean $E(X)$.

Solution

Finding a:

For a valid pdf, we need $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$:

$$\begin{aligned}
\int_{-1}^{0} ax\,dx + \int_{0}^{1} x\,dx &= 1 \\
a\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} &= 1 \\
a\left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) &= 1 \\
-\frac{a}{2} + \frac{1}{2} &= 1 \\
-\frac{a}{2} &= \frac{1}{2} \\
a &= -1
\end{aligned}$$

Therefore:

$$f_X(x) = \begin{cases} -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

CDF $F_X(x)$:

For $x < -1$: $F_X(x) = 0$

For $-1 \le x < 0$:

$$\begin{aligned}
F_X(x) &= \int_{-1}^{x} (-t)\,dt = -\left[\frac{t^2}{2}\right]_{-1}^{x} \\
&= -\frac{x^2}{2} + \frac{1}{2} = \frac{1 - x^2}{2}
\end{aligned}$$

For $0 \le x < 1$:

$$\begin{aligned}
F_X(x) &= \int_{-1}^{0} (-t)\,dt + \int_{0}^{x} t\,dt \\
&= \frac{1}{2} + \left[\frac{t^2}{2}\right]_{0}^{x} = \frac{1}{2} + \frac{x^2}{2} = \frac{1 + x^2}{2}
\end{aligned}$$

For $x \ge 1$: $F_X(x) = 1$

Therefore:

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{1-x^2}{2}, & -1 \le x < 0 \\ \frac{1+x^2}{2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

Quantile Function $\tau_{X,q}$:

For $0 < q < 0.5$: Solve $\frac{1-\tau_{X,q}^2}{2} = q$

$$\begin{aligned} 1 - \tau_{X,q}^2 &= 2q \\ \tau_{X,q}^2 &= 1 - 2q \\ \tau_{X,q} &= -\sqrt{1-2q} \quad (\text{negative since } -1 \le \tau_{X,q} < 0) \end{aligned}$$

For $0.5 \le q < 1$: Solve $\frac{1+\tau_{X,q}^2}{2} = q$

$$\begin{aligned} 1 + \tau_{X,q}^2 &= 2q \\ \tau_{X,q}^2 &= 2q - 1 \\ \tau_{X,q} &= \sqrt{2q-1} \quad (\text{positive since } 0 \le \tau_{X,q} < 1) \end{aligned}$$

Therefore:

$$\tau_{X,q} = \begin{cases} -\sqrt{1-2q}, & 0 < q < 0.5 \\ \sqrt{2q-1}, & 0.5 \le q < 1 \end{cases}$$

Population Mean $E(X)$:

$$\begin{aligned} E(X) &= \int_{-1}^{0} x \cdot (-x)\, dx + \int_{0}^{1} x \cdot x\, dx \\ &= \int_{-1}^{0} (-x^2)\, dx + \int_{0}^{1} x^2\, dx \\ &= -\left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1} \\ &= -\left(0 - \left(-\frac{1}{3}\right)\right) + \left(\frac{1}{3} - 0\right) \\ &= -\frac{1}{3} + \frac{1}{3} = 0 \end{aligned}$$

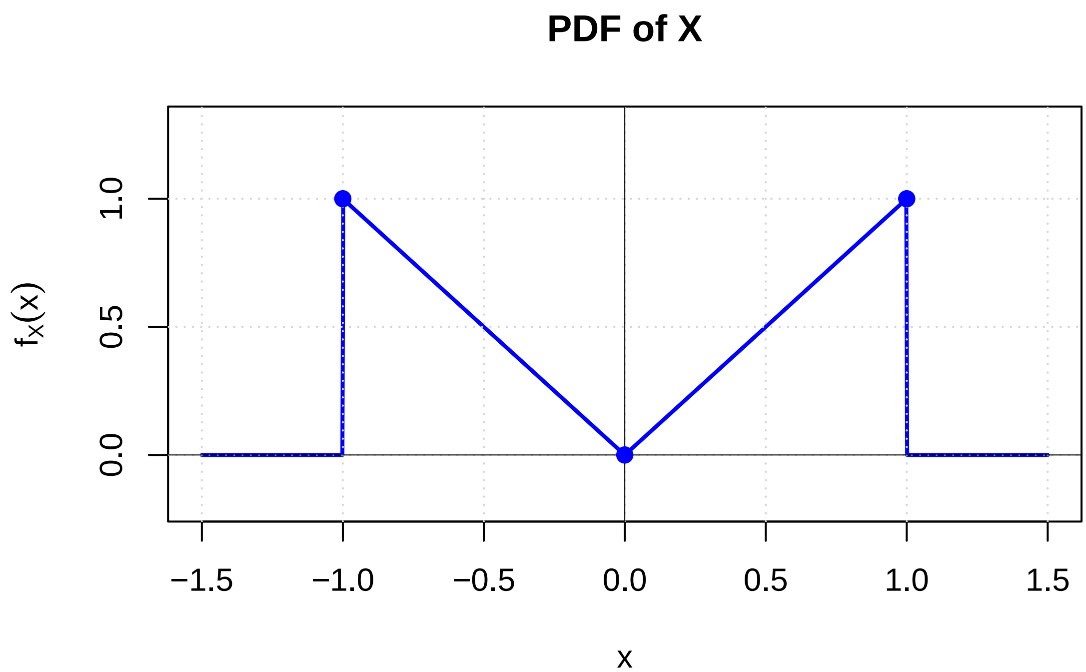

Figure 3: PDF of Random Variable X

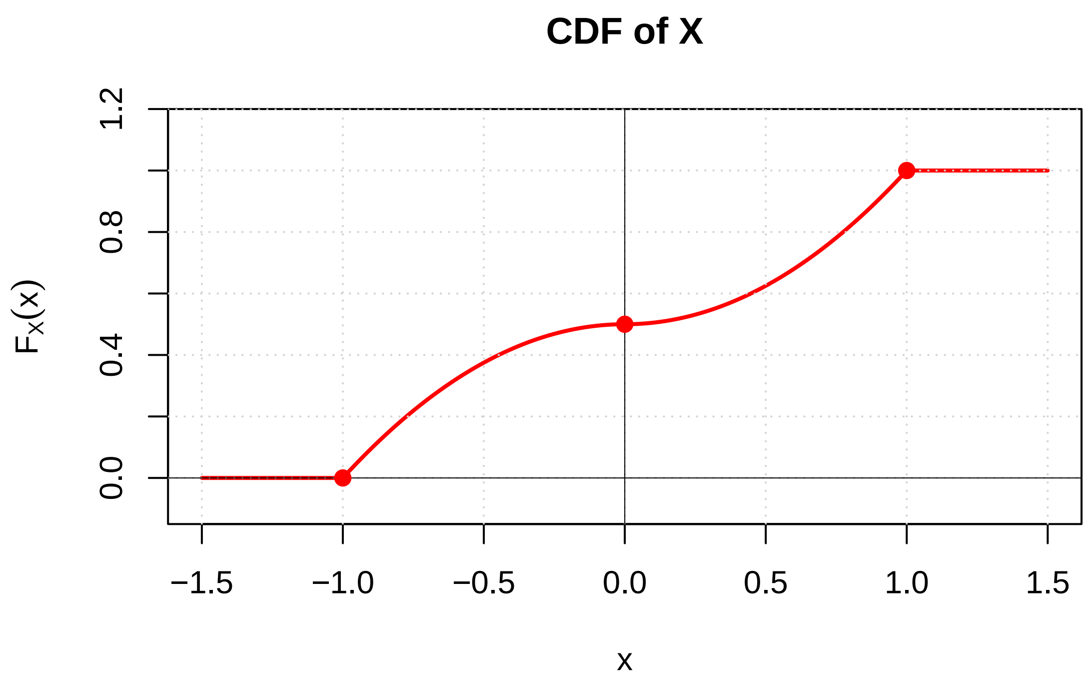

Figure 4: CDF of Random Variable X

PDF and CDF for Exercise 2

Applied Exercise 3

Problem: Random variables X and Y have the following joint pdf:

$$f_{XY}(x,y) = \begin{cases} axy, & 0 \le x < y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine: (a) the value of a, (b) $P(X \le 2Y)$, (c) marginal pdf $f_X(x)$, (d) $E(X)$, (e) $\text{Var}(X)$, (f) $E(Y|X = 0.5)$, and (g) check if X and Y are independent.

Solution

(a) Finding a**:**

The region of integration is $\{(x, y) : 0 \le x < y \le 1\}$.

$$\begin{aligned}
\int_0^1 \int_x^1 axy\,dy\,dx &= 1 \\
a \int_0^1 x \left[\frac{y^2}{2}\right]_x^1 dx &= 1 \\
a \int_0^1 x \left(\frac{1}{2} - \frac{x^2}{2}\right) dx &= 1 \\
\frac{a}{2} \int_0^1 (x - x^3)\,dx &= 1 \\
\frac{a}{2} \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 &= 1 \\
\frac{a}{2} \left(\frac{1}{2} - \frac{1}{4}\right) &= 1 \\
\frac{a}{2} \cdot \frac{1}{4} &= 1 \\
a &= 8
\end{aligned}$$

(b) Finding $P(X \le 2Y)$**:**

The condition $X \le 2Y$ is equivalent to $Y \ge \frac{X}{2}$.

Given the constraint $0 \le x < y \le 1$, we need $y \ge \max(x, \frac{x}{2}) = x$ (since $x > \frac{x}{2}$ for $x > 0$).

Therefore, $X \leq 2Y$ is always satisfied in the support region, so:

$$P(X \leq 2Y) = 1$$

(c) Marginal pdf $f_X(x)$:

For $0 \leq x \leq 1$:

$$\begin{aligned} f_X(x) &= \int_x^1 8xy\,dy = 8x\left[\frac{y^2}{2}\right]_x^1 \\ &= 8x\left(\frac{1}{2} - \frac{x^2}{2}\right) = 4x(1 - x^2) \end{aligned}$$

Therefore:

$$f_X(x) = \begin{cases} 4x(1 - x^2), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(d) Expected Value $E(X)$:

$$\begin{aligned} E(X) &= \int_0^1 x \cdot 4x(1 - x^2)\,dx = 4\int_0^1 (x^2 - x^4)\,dx \\ &= 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) \\ &= 4 \cdot \frac{2}{15} = \frac{8}{15} \end{aligned}$$

(e) Variance $\mathrm{Var}(X)$:

First, compute $E(X^2)$:

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 \cdot 4x(1 - x^2)\,dx = 4\int_0^1 (x^3 - x^5)\,dx \\ &= 4\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 4\left(\frac{1}{4} - \frac{1}{6}\right) \\ &= 4 \cdot \frac{1}{12} = \frac{1}{3} \end{aligned}$$

Therefore:

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - [E(X)]^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 \\ &= \frac{1}{3} - \frac{64}{225} = \frac{75}{225} - \frac{64}{225} = \frac{11}{225} \end{aligned}$$

(f) Conditional Expected Value $E(Y|X = 0.5)$:

First, find the conditional pdf $f_{Y|X}(y|x = 0.5)$:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

For $x = 0.5$:

$$f_{Y|X}(y|0.5) = \frac{2y}{1-0.25} = \frac{2y}{0.75} = \frac{8y}{3}, \quad 0.5 < y \leq 1$$

Then:

$$\begin{aligned} E(Y|X = 0.5) &= \int_{0.5}^{1} y \cdot \frac{8y}{3}\, dy = \frac{8}{3}\int_{0.5}^{1} y^2\, dy \\ &= \frac{8}{3}\left[\frac{y^3}{3}\right]_{0.5}^{1} = \frac{8}{9}\left(1 - \frac{1}{8}\right) \\ &= \frac{8}{9} \cdot \frac{7}{8} = \frac{7}{9} \end{aligned}$$

(g) Independence Check:

For independence, we need $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

First, find $f_Y(y)$. For $0 \leq y \leq 1$:

$$\begin{aligned} f_Y(y) &= \int_0^y 8xy\, dx = 8y\left[\frac{x^2}{2}\right]_0^y \\ &= 8y \cdot \frac{y^2}{2} = 4y^3 \end{aligned}$$

Now check: $f_X(x) \cdot f_Y(y) = 4x(1-x^2) \cdot 4y^3 = 16xy^3(1-x^2)$

This does not equal $f_{XY}(x,y) = 8xy$.

Conclusion: X and Y are **NOT independent**.

Support Region for Exercise 3

The support region where $f_{XY}(x,y) > 0$ is the triangular region defined by $0 \leq x < y \leq 1$.

The shaded triangular region represents the area where the joint pdf is positive. The region is bounded by:

- Lower boundary: the line $y = x$
- Upper boundary: the line $y = 1$
- Left boundary: the line $x = 0$

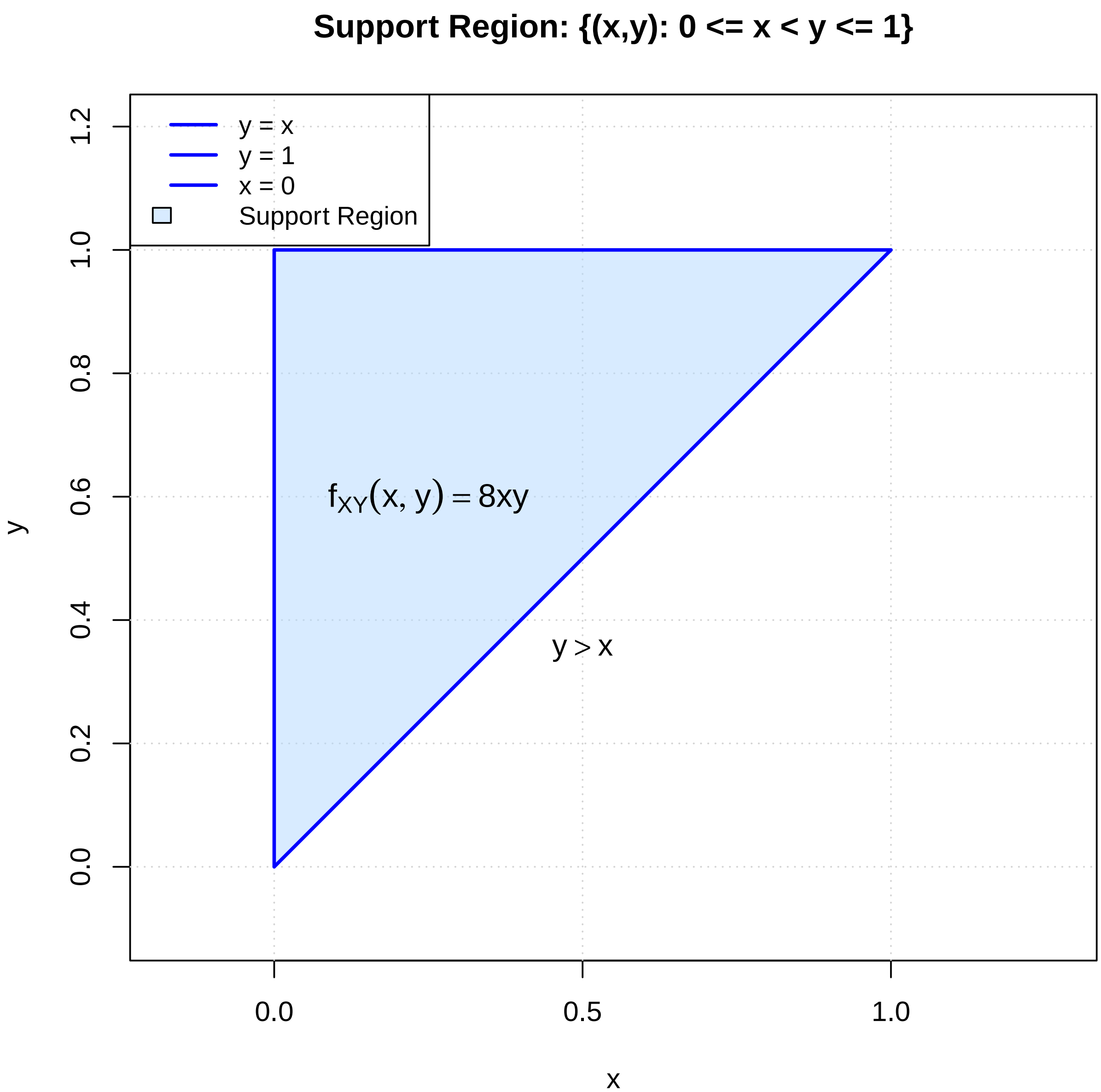

Figure 5: Support Region where joint PDF is positive