AE Lecture 4

Counting Methods

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Reference

These problems is from JA Chapter 3 Exercise 12, Chapter Exercise 3 and 5

Application Exercise 1

12. Suppose that on any given weekday, 70% of college students eat breakfast, 60% do homework, and 85% do at least one of these two things.

    a. Are the events "Student eats breakfast" and "Student does homework" independent?

       Solution: Let B and H denote the events of a student eating breakfast and doing homework. Recall

       $$P(B \cap H) = P(B) + P(H) - P(B \cup H) = 0.7 + 0.6 - 0.85 = 0.45.$$

       This means B and H are not independent since $P(B \cap H) \neq P(B)P(H)$

    b. If a randomly selected student eats breakfast, what is the probability that they do homework?

       Solution:

       $$P(H \mid B) = \frac{P(B \cap H)}{P(B)} = 0.45/0.7 = 9/14 \approx 0.6249.$$

c. Suppose that two students are selected at random and their behaviors are independent of each other. What is the probability that exactly one of the two students does homework?

Solution:

$$P(\{\text{student 1 does homework} \cap \text{student 2 doesn't}\} \cup \{\text{student 2 does homework} \cap \text{student 1 doesn't}\})$$

$$= P(\text{student 1 does homework})P(\text{student 2 doesn't}) + P(\text{student 2 does homework})P(\text{student 1 does}$$

$$= 0.6 \cdot 0.4 + 0.6 \cdot 0.4 = 0.48.$$

d. Suppose that 80% of students who eat breakfast on a given day also eat lunch that day, while 90% of students who don't eat breakfast on a given day eat lunch that day. For a student who eats lunch on a given day, what is the probability that she or he eats breakfast on that day?

Solution:

Let L denote the event of a student eating lunch. From Bayes' Theorem,

$$P(B \mid L) = \frac{P(L \mid B)P(B)P(B)}{P(L \mid B)P(B)P(B) + P(L \mid B^c)P(B^c)} = \frac{(0.8)(0.7)}{(0.8)(0.7) + (0.9)(0.3)} = \frac{0.56}{0.83} \approx 0.6747.$$

Application Exercise 2

3. A company visits a college campus to interview students. The company has seven economics majors, six finance majors,and five accounting majors from which to choose. Unfortunately, the company has lost everyone's résumés, so they randomly pick three students to interview.

   a. What is the probability that all three interviewees are economics students?

   Solution:

   The probability of three Economics interviewees is:

$$\frac{7}{18}\frac{6}{17}\frac{5}{16} \approx 0.0429.$$

   b. What is the probability that all three interviewees are from the same major?

   Solution:

   The probability that all interviewees are the same major is the sum of three probabilities corresponding to each major (since they are disjoin events):

$$\frac{7}{18}\frac{6}{17}\frac{5}{16} + \frac{6}{18}\frac{5}{17}\frac{4}{16} + \frac{5}{18}\frac{4}{17}\frac{3}{16} \approx 0.0797$$

c. What is the probability that the set of three interviewees has either no economics students, or no Finance students?

Solution:

Let A be the event that no Economics students are interviewed and B be the event that no Finance students are interviewed. Then, the probability of A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{11}{18}\frac{10}{17}\frac{9}{16} + \frac{12}{18}\frac{11}{17}\frac{10}{16} - \frac{5}{18}\frac{4}{17}\frac{3}{16} \approx 0.4596$$

d. What is the probability that at least one of the majors has no students interviewed?

Solution:

Let C be the event that no Accounting students are interviewed. Then the probability of A or B or C is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(C \cap C) + P(A \cap B \cap C).$$

We have $P(A),P(B),P(A B) from previous part. The other probabilities are $P(C) = \frac{13}{18}\frac{12}{17}\frac{11}{16}$, $P(A \cap C) = \frac{6}{18}\frac{5}{17}\frac{4}{16}$, $P(B \cap C) = \frac{7}{18}\frac{6}{17}\frac{5}{16}$, and $P(A \cap B \cap C) = 0$. Substitute all this probability value we get that $P(A \cup B \cup C) = \frac{36363}{18 \cdot 17 \cdot 16} = 0.7426$.

Application Exercise 3

5. An office manager needs to assign offices to nine employees. There are three offices available: office A, which holds two workers; office B, which holds three workers; and office C, which holds four workers.

   a. How many possible office assignments of the employees are there?

   Solution:

   There are $\binom{9}{2, 3, 4} = \frac{9!}{2!3!4!} = 1260$ possible office assignments.

   b. McKenna is one of the nine workers. If assignments are made at random, what is the probability that McKenna gets assigned to office C?

   Solution:

   McKenna has an equal chance of being in any of the 9 office slots, of which 4 are in office C, so the probability is $\frac{4}{9}$.

   c. If assignments are made at random, what is the probability that McKenna is in the same office as her coworker Daniel?

   Solution:

   The probability that McKenna and Daniel are in the same office is the sum of three probabilities: the probability that both are in office A, the probability that both are in office B, and the probability that both are in office C:

$$\frac{2}{9}\frac{1}{8}+\frac{3}{9}\frac{2}{8}+\frac{4}{9}\frac{3}{8}=\frac{20}{72}\approx 0.2778.$$

d. If assignments are made at random, what is the probability that McKenna is in the same office as her coworkers Daniel and Alma?

Solution:

Similar to previous part, the probability that McKenna is in the same office as both Daniel and Alma is:

$$\frac{2}{9}\frac{1}{8}\frac{0}{78}+\frac{3}{9}\frac{2}{8}\frac{1}{7}+\frac{4}{9}\frac{3}{8}\frac{2}{8}=\frac{30}{504}\approx 0.0595.$$